JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 47, 58–68 (1974)

On Positive Solutions of Perturbed Nonlinear Differential Equations

ATHANASSIOS G. KARTSATOS

Department of Mathematics, University of South Florida, Tampa, Florida 33620

Submitted by Alex McNabb

Some results are given concerning positive solutions of equations of the form

$$x^{(n)} + P(t)\,G(x) = Q(t, x).$$

Let class I (II) consist of all n-times differentiable functions $x(t)$, such that $x(t) > 0$ and $x^{(n-1)}(t) \geqslant 0$ ($x^{(n-1)}(t) \leqslant 0$) for all large t. Two theorems are given guaranteeing the nonexistence of solutions in class I and II, respectively, and three theorems ensure the convergence to zero of positive solutions. A recent result of Hammett concerning the second-order case is extended to the general case.

1. INTRODUCTION

The objective of this paper is to give some results concerning the asymptotic behavior of positive solutions of the equation

$$x^{(n)} + P(t)\,G(x) = Q(t, x) \qquad \text{for } n \text{ even}, \tag{$*$}$$

where, P, G, and Q are real valued and continuous on $[0, \infty)$, $(-\infty, \infty)$, and $[0, \infty) \times (-\infty, \infty)$, respectively. In [3] we gave a first classification of the positive solutions of $(*)$ in the case where $P(t)$ is not assumed to be positive for all large t. Namely, we defined four classes of positive functions, and we examined the existence or nonexistence of solutions of $(*)$ in those classes. The present paper can be considered a continuation of [3]. It is to be understood that corresponding results hold for the negative solutions of $(*)$ if we slightly modify the hypotheses of the results of this paper.

In Section 2 we give some preliminary facts and a lemma which extends to the general case of Lemma 2.1 in [7]. In Section 3 we define (as in [3]) two classes of functions (I and II) and we give two theorems (Theorems 1 and 2) which guarantee the nonexistence of solutions of $(*)$ in class I and II, respectively. In Section 4 we present conditions under which certain, or all, positive solutions of $(*)$ converge to zero as $t \to \infty$. Some examples and indications of possible extensions are given at the end of the paper.

2. Preliminaries

Throughout the sequel, all functions considered will be assumed continuous on their domains, and the existence of solutions of $(*)$, which are valid for all large t will be assumed without further mention.

A solution of $(*)$ is a function $x \in C^n([t_x, \infty), R)$, which satisfies $(*)$ on $[t_x, \infty)$. Here t_x depends on the particular solution x and $R = (-\infty, \infty)$. Denote by U the family of all such solutions of $(*)$. A function $x \in U$ is said to be "oscillatory" if there exists a sequence $\{t_n\}$, $n = 1, 2, \ldots$, such that $t_n \geqslant t_x$, $\lim_{n\to\infty} t_n = +\infty$, and $x(t_n) = 0$.

Lemma 1. *Assume that for a function* $L\colon [t_0, \infty) \to R$ $(t_0 > 0)$ *we have*

$$\int_{t_0}^{t} s^k L(s)\, ds \geqslant 0 \qquad \textit{for every } t \geqslant t_0,$$

where k is a nonnegative integer. Then

$$\int_{t_0}^{t} s^m L(s)\, ds \geqslant 0$$

for every nonnegative integer $m \leqslant k - 1$.

Proof. Let m $(0 \leqslant m \leqslant k-1)$ be an integer. Then we have

$$\begin{aligned} 0 \leqslant \int_{t_0}^{t} s^k L(s)\, ds &= \int_{t_0}^{t} s^{k-m} \left[\int_{t_0}^{s} v^m L(v)\, dv\right]' ds \\ &= t^{k-m} \int_{t_0}^{t} s^m L(s)\, ds - (k-m) \int_{t_0}^{t} s^{k-m-1} \int_{t_0}^{s} v^m L(v)\, dv\, ds. \end{aligned} \tag{1}$$

Define

$$P^*(t) = \int_{t_0}^{t} s^{k-m-1} \int_{t_0}^{s} v^m L(v)\, dv\, ds.$$

Then (1) yields

$$tP^{*\prime}(t) - (k-m)\, P^*(t) \geqslant 0, \qquad t \geqslant t_0, \tag{2}$$

which in turn implies

$$\frac{d}{dt}\left(\frac{P^*(t)}{t^{k-m}}\right) \geqslant 0, \qquad t \geqslant t_0. \tag{3}$$

Since $P^*(t_0)/t_0^{k-m} = 0$, it follows that $P^*(t) \geqslant 0$ for $t \geqslant t_0$; from (1) we obtain now

$$\int_{t_0}^{t} s^m L(s)\, ds \geqslant 0, \qquad t \geqslant t_0. \tag{4}$$

3. Classes I and II

Let us consider the following two classes.

Class I. All n-times differentiable functions $x(t)$ such that $x(t) > 0$ and $x^{(n-1)}(t) \geqslant 0$ for all large t.

Class II. All n-times differentiable functions $x(t)$ such that $x(t) > 0$ and $x^{(n-1)}(t) \leqslant 0$ for all large t.

Theorem 1. *Assume that k is a nonnegative integer and*

(i) $P\colon [0, \infty) \to R$ *and* $\int_0^\infty t^k P(t)\, dt = +\infty$;

(ii) $G\colon R \to R$, $G(x) > 0$ *for* $x > 0$, *increasing on* $(0, \infty)$, *and*

$$\int_\epsilon^\infty [G(s)]^{-1}\, ds < +\infty \qquad \textit{for some } \epsilon > 0;$$

(iii) $Q\colon [0, \infty) \times R \to R$ *and* $\int_0^\infty t^k Q_0(t)\, dt < +\infty$, *where*

$$| Q(t, u)/G(u)| \leqslant Q_0(t) \qquad \textit{for every } (t, u) \in [0, \infty) \times (0, \infty).$$

Then there is no $x \in U$ which belongs to class I.

Proof. Assume that $x \in U$ is such that $x(t) > 0$ and $x^{(n-1)}(t) \geqslant 0$ for all $t \in [t_1, \infty)$, $t_1 \geqslant t_x$. Then it can be easily proved that $x'(t) \geqslant 0$, $t \in [t_1, \infty)$ (cf., e.g., [3]–[6]). Thus, by multiplication of $(*)$ by $t^k/G(x(t))$ and then integration from t_1 to $t \geqslant t_1$, we obtain, for $k \geqslant 1$,

$$\begin{aligned} tP_0'(t) - kP_0(t) \leqslant t_1 P_0'(t_1) - \int_{t_1}^t s^k P(s)\, ds \\ + \int_{t_1}^t s^k x^{(n-1)}(s)\, d[1/G(x(s))] + \int_{t_1}^t s^k Q_0(s)\, ds, \end{aligned} \tag{5}$$

where

$$P_0(t) = \int_{t_1}^t [s^{k-1} x^{(n-1)}(s)/G(x(s))]\, ds. \tag{6}$$

Since the second (Riemann–Stieltjes) integral in the right-hand side of (5) is nonpositive $[1/G(x(t))$ is decreasing$]$ and the last integral converges, it follows that $\lim_{t\to\infty} P_0(t) = +\infty$, which implies

$$\int_{t_1}^\infty [t^{n-2} x^{(n-1)}(t)/G(x(t))]\, dt = +\infty, \tag{7}$$

and the proof follows as in Theorem 2 in [4], by working with Riemann–Stieltjes integrals.

The case $k = 0$ is actually Theorem 2.1 in [3]. In [3] it has been assumed that $G'(x) \geqslant 0$; this assumption is avoided here by use of Riemann–Stieltjes integrals.

THEOREM 2. *Assume that*

(i) $P\colon [0, \infty) \to R$ *and*

(ii) $G\colon R \to R$ *and there exists a solution* $x \in U$ *such that, for some non-negative integer* k,

$$\int_0^\infty s^k[P(s)\, G(x(s)) - Q(s, x(s))]\, ds = +\infty.$$

Then $x(t)$ *cannot belong to class* II.

Proof. Assume that $x(t) > 0$ and $x^{(n-1)}(t) \leqslant 0$ for every $t \geqslant t_1$, $t_1 \geqslant t_x$. Then it is easy to show (see, for example, Bhatia [1] and Travis [7]) that there exists $t_2 \geqslant t_1$ with the property

$$\int_{t_2}^t s^k F(s, x(s))\, ds \geqslant 0 \qquad \text{for every } t \geqslant t_2, \tag{8}$$

where $F \equiv PG - Q$. If $k = 0$, then the theorem follows by integration of $(*)$ and then taking the limit as $t \to \infty$. Suppose that $k \geqslant 1$. Differentiation of the function $t^k x^{(n-1)}(t)$ and then integration from t_2 to $t \geqslant t_2$ yields

$$t^k x^{(n-1)}(t) - k \int_{t_2}^t s^{k-1} x^{(n-1)}(s)\, ds = t_2{}^k x^{(n-1)}(t_2) - \int_{t_2}^t s^k F(s, x(s))\, ds \leqslant 0. \tag{9}$$

Thus, $(d/dt)\,[P^*(t)/t^k] \leqslant 0$, where

$$P^*(t) = \int_{t_2}^t s^{k-1} x^{(n-1)}(s)\, ds.$$

Since $P^*(t_2)/t_2{}^k = 0$ (we can assume that $t_2 > 0$), it follows that $P^*(t) \leqslant 0$ for all $t \in [t_2, \infty)$. This combined with (9) gives $\lim_{t\to\infty} t^k x^{(n-1)}(t) = -\infty$. Consequently, $x^{(n-1)}(t) < 0$ for every $t \geqslant t_3 \geqslant t_2$. Now, it is obvious that we can find a number $t_4 \geqslant t_3$ such that (8) holds with t_2 replaced by t_4. By Lemma 1,

$$\int_{t_4}^t F(s, x(s))\, ds \geqslant 0 \qquad \text{for every } t \geqslant t_4,$$

and an integration of $(*)$ gives

$$x^{(n-1)}(t) = x^{(n-1)}(t_4) - \int_{t_4}^{t} F(s, x(s))\, ds \leqslant x^{(n-1)}(t_4) < 0, \tag{10}$$

which implies $\lim_{t\to\infty} x(t) = -\infty$, a contradiction.

4. The Convergence to Zero of Positive Solutions of $(*)$

Theorem 3. *Assume that P satisfies* (i) *of Theorem* 1 *with an integer $k \geqslant 0$ not necessarily bounded above by $n - 1$. Assume that G is as in* (ii) *of Theorem* 1 *without (necessarily) the integrability condition. Let $Q \equiv 0$. Then every bounded solution of $(*)$ that belongs to class* II, *must satisfy* $\lim_{t\to\infty} x(t) = 0$.

Proof. Suppose that $x \in U$ is such that $x(t) > 0$ and $x^{(n-1)}(t) \leqslant 0$ for all large t. Then if $x(t)$ is bounded, $x'(t) \leqslant 0$ for all large t. In fact, this follows from the fact that $n - 1$ is odd and that all the derivatives of $x(t)$, up to the order $n - 1$, are eventually of constant sign. Now, let $x(t) > 0$, $x^{(n-1)}(t) \leqslant 0$, and $x'(t) \leqslant 0$ for all $t \in [t_1, \infty)$, $t_1 \geqslant t_x$. Let $k \geqslant 1$, and let t_2 be such that $t_2 \geqslant t_1$ and

$$\int_{t_2}^{t} s^k P(s)\, ds \geqslant 0, \qquad t \in [t_2, \infty).$$

Then from (5) we obtain

$$tP_0'(t) - kP_0(t) \leqslant t_2 P_0'(t_2) - \int_{t_2}^{t} s^k P(s)\, ds, \tag{11}$$

where $P_0(t)$ denotes now the function $\int_{t_2}^{t} [s^{k-1} x^{(n-1)}(s)/G(x(s))]\, ds$. From (11), by taking the limit as $t \to \infty$, we obtain $x^{(n-1)}(t) < 0$ for all $t \geqslant$ (some) $t_3 \geqslant t_2$.

By Lemma 1, there exists $t_4 \geqslant t_3$ such that

$$\int_{t_4}^{t} P(s)\, ds \geqslant 0 \qquad \text{for all } t \in [t_4, \infty). \tag{12}$$

Thus, we have

$$\begin{aligned} x^{(n-1)}(t)/G(x(t)) &\leqslant x^{(n-1)}(t_4)/G(x(t_4)) - \int_{t_4}^{t} P(s)\, ds \\ &\leqslant x^{(n-1)}(t_4)/G(x(t_4)) < 0. \end{aligned} \tag{13}$$

This is easily obtained from $(*)$ by dividing by $G(x(t))$ and then integrating from t_4 to $t \geqslant t_4$. Now assume that $\lim_{t\to\infty} x(t) = r > 0$. Then from (13) we obtain

$$x^{(n-1)}(t) \leqslant [x^{(n-1)}(t_4)/G(x(t_4))]\, G(r) < 0 \tag{14}$$

for every $t \geqslant t_4$. Since (14) implies $\lim_{t\to\infty} x(t) = -\infty$, which is a contradiction, our assertion is true in the case $k \geqslant 1$. The case $k = 0$ can be treated similarly by using (13) instead of (11).

More satisfactory results can be obtained if P can be written as a sum of two functions P_1 and P_2, where P_1 is "large" and positive and P_2G is "small."

THEOREM 4. *Suppose that* (i) $P = P_1 + P_2$ *where* $P_1\colon [0, \infty) \to (0, \infty)$, $P_2\colon [0, \infty) \to R$ *and*

$$\int_{t_1}^{\infty} t^{n-1} P_1(t)\, dt = +\infty, \qquad \int_{t_1}^{\infty} t^{n-1} \,|\, P_2(t)|\, G(x(t))\, dt < +\infty,$$

where $x \in U$ *is such that* $x(t) > 0$ *for all* $t \geqslant t_1 \geqslant t_x$: (ii) $G\colon R \to R$ *is increasing on* R, $G(x) > 0$ *for* $x > 0$, *and*

(ii′) $$\int_{\epsilon}^{\infty} [G(s)]^{-1}\, ds < +\infty \qquad \textit{for some } \epsilon > 0;$$

(iii) $|\, Q(t, x(t))| \leqslant Q_1(t)$, *where* $Q_1\colon [0, \infty) \to [0, \infty)$ *and*

$$\int_{t_1}^{\infty} t^{n-1} Q_1(t)\, dt < +\infty.$$

Then $\lim_{t\to\infty} x(t) = 0$.

Proof. Consider the function

$$S(t) \equiv [1/(n-1)!] \int_{t}^{\infty} (t-s)^{n-1}\, [P_2(s)\, G(x(s)) - Q(s, x(s))]\, ds \tag{15}$$

for every $t \geqslant t_1$. S is well defined because

$$|\, S(t)| \leqslant \int_{t}^{\infty} (t - t_1 + 1)^{n-1}\, [|\, P_2(s)|\, G(x(s)) + Q_1(s)]\, ds < +\infty. \tag{16}$$

$S(t)$ is continuously differentiable on $[t_1, \infty)$ and

$$S^{(n)}(t) = -\, P_2(t)\, G(x(t)) + Q(t, x(t)).$$

It follows from Theorem 3 in [5] that $\lim_{t\to\infty} x(t) = 0$.

In the proof of the next theorem, which extends the main result of Hammett [8], we make use of the following.

LEMMA 2. *Suppose that conditions* (i) *and* (iii) *of Theorem* 4 *are satisfied with* $n-1$ *replaced by* k, *where* k *is an integer with* $0 \leqslant k \leqslant n-2$. *Moreover, assume that* (ii) *of the same theorem is satisfied* (*except maybe* (ii′)), *and let*

$$\liminf_{u\to\infty} G(u)/u > 0.$$

Then

$$\int_{t_1}^{\infty} t^k P_1(t)\, G(x(t))\, dt < +\infty.$$

Proof. Let $k = 0$. Then, integrating $(*)$ from t_1 to $t \geqslant t_1$, we obtain

$$x^{(n-1)}(t) = x^{(n-1)}(t_1) - \int_{t_1}^{t} P_1(s)\, G(x(s))\, ds + \int_{t_1}^{t} T_2(s)\, ds, \tag{17}$$

where $T_2(t) \equiv [Q - P_2 G]\,(t)$. Since the last integral in (17) converges and the integrand in the first one is positive, it follows that $\lim_{t\to\infty} x^{(n-1)}(t) = \lambda_1$ (finite or $-\infty$). If $\lambda_1 = -\infty$, then $x(t) \to -\infty$ as $t \to \infty$, which is contradiction. Thus, λ_1 is finite and our assertion follows for $k = 0$. From (17) we actually obtain $\lambda_1 = 0$.

In fact, assume first that $\lambda_1 < 0$. Then, since $x^{(n-1)}(t)$ is eventually bounded above by a negative constant, it follows that $\lim_{t\to\infty} x(t) = -\infty$, which is a contradiction. Now suppose that $\lambda_1 > 0$. Then $x^{(n-2)}(t) \geqslant \lambda_2 t$ for some positive $\lambda_2 < 1$, and finally, $x(t) \geqslant \lambda_n t^{n-1}$ for all large t $(0 < \lambda_n < 1)$. Let $K > 0$ and $L > 0$ be such that $G(x)/x \geqslant K$ for $x \geqslant L$. Moreover, choose t_1 so large that $L < \lambda_n t_1^{n-1}$.

Then, since $G(x(t)) \geqslant G(\lambda_n t^{n-1}) \geqslant K\lambda_n t^{n-1}$ for all $t \geqslant t_1$, it follows from (17) that

$$x^{(n-1)}(t) \leqslant x^{(n-1)}(t_1) - K\lambda_n \int_{t_1}^{t} s^{n-1} P_1(s)\, ds + \int_{t_1}^{t} T_2(s)\, ds; \tag{18}$$

that is

$$\lim_{t\to\infty} x^{(n-1)}(t) = -\infty,$$

which is a contradiction. Thus,

$$\lim_{t\to\infty} x^{(n-1)}(t) = 0,$$

which combined with (17) gives

$$x^{(n-1)}(t) = \int_{t}^{\infty} P_1(s)\, G(x(s))\, ds - \int_{t}^{\infty} T_2(s)\, ds, \qquad t \geqslant t_1. \tag{19}$$

Now integrate (19) to obtain

$$x^{(n-2)}(t) = x^{(n-2)}(t_1) + \int_{t_1}^{t}\int_{s}^{\infty} P_1(u)\, G(x(u))\, du\, ds - \int_{t_1}^{t}\int_{s}^{\infty} T_2(u)\, du\, ds \tag{20}$$

for every $t \geqslant t_1$, and let $k \geqslant 1$. Then we have

$$\int_{t_1}^{t}\int_{s}^{\infty} F(u)\, du\, ds = \int_{t_1}^{t} H'(s)\, ds = H(t) - H(t_1), \tag{21}$$

where

$$F(u) = P_1(u)\, G(x(u)) \qquad \text{and} \qquad H(s) = \int_{s}^{\infty} (s-u)\, F(u)\, du.$$

$H(t)$ is bounded by assumption. A corresponding identity holds with respect to $T_2(u)$. Thus, from (20) we obtain

$$\begin{aligned} x^{(n-2)}(t) = x^{(n-2)}(t_1) &+ \int_{t}^{\infty} (t-s)\, P_1(s)\, G(x(s))\, ds \\ &+ \int_{t}^{\infty} (t-s)\, T_2(s)\, ds + Z(t_1), \end{aligned} \tag{22}$$

where $Z(t_1)$ equals minus the sum of the above integrals evaluated at t_1. Now it is easy to see that $\lim_{t\to\infty} x^{(n-2)}(t)$ cannot be different from zero because, if this limit is negative, then $\lim_{t\to\infty} x(t) = -\infty$ and, if it is positive, then $x(t) \geqslant pt^{n-2}$ (for some p with $0 < p < 1$) for all $t \geqslant$ (a suitably chosen) t_1, which implies a contradiction as above. Consequently, taking the limit of (22) as $t \to \infty$, we obtain $x^{(n-2)}(t_1) = -Z(t_1)$ and, since this holds if t_1 is replaced by any $t \geqslant t_1$,

$$x^{(n-2)}(t) = \int_{t}^{\infty} (t-s)\, P_1(s)\, G(x(s))\, ds + \int_{t}^{\infty} (t-s)\, T_2(t)\, ds. \tag{23}$$

Similarly, by induction, we obtain

$$x^{(n-k-1)}(t) = (1/k!) \left[\int_{t}^{\infty} (t-s)^k\, P_1(s)\, G(x(s))\, ds - \int_{t}^{\infty} (t-s)^k\, T_2(s)\, ds\right], \tag{24}$$

where both integrals converge. This completes the proof.

Now we are ready for the following.

THEOREM 5. *In Lemma* 2, *assume further that* $P_1(t) \geqslant M/t^k$ $(M > 0)$ *for all large* t. *Then* $\lim_{t\to\infty} x(t) = 0$.

Proof. Obviously, $\lim_{t\to\infty} \inf x(t) = 0$; otherwise, $x(t) \geqslant N > 0$ for all large t implies

$$\int^{\infty} s^k P_1(s)\, G(x(s))\, ds \geqslant \int^{\infty} s^k P_1(s)\, G(N)\, ds = +\infty. \tag{25}$$

Now let $\bar{t}$ be large enough so that $P(t) \geqslant M/t^k$ and (24) hold for all $t \geqslant \bar{t}$.

Assume that $\lim_{t\to\infty} \sup x(t) = N_1 > 0$. Then, following Hammett [8], there exists a sequence $\{t_n\}$ for $t_n \geqslant \bar{t}$ and $n \geqslant 0$, such that

(a) $\lim_{n\to\infty} t_n = +\infty$,

(b) $x(t_n) \geqslant N_1$ for each n, and

(c) for each $n \geqslant 1$ there exists t_n' such that $t_{n-1} < t_n' < t_n$ and $x(t_n') < N_1/2$.

It follows that there exists a sequence $[a_n, b_n]$ such that $b_n - a_n \geqslant N_2 > 0$ (N_2 is a positive constant) for every $n \geqslant 1$ and $G(x(t)) \geqslant G(N_1/2) > 0$ on $[a_n, b_n]$.

Now

$$\int_{\bar{t}}^{\infty} t^k P_1(t)\, G(x(t))\, dt < +\infty \tag{26}$$

and

$$\int_{a_1}^{b_m} t^k P_1(t)\, G(x(t))\, dt > \sum_{j=1}^{m} \int_{a_j}^{b_j} t^k P_1(t)\, G(x(t))\, dt > MG(N_1/2)\, N_2 m \to +\infty, \tag{27}$$

as $m \to \infty$, which contradicts (26).

A simple consequence of the above theorem is the following.

COROLLARY 1. *If in the differential equation*

$$x^{(n)} + P(t)\, G(x) = Q(t), \qquad \textit{where} \qquad n \textit{ is even}, \tag{28}$$

(i) $P(t) \geqslant M/t^k$, *for all large* t, k *a nonnegative integer*, $0 \leqslant k \leqslant n-2$;

(ii) $\int_0^\infty t^k\, |Q(t)|\, dt < +\infty$;

(iii) G *as in Lemma* 2.

Then every eventually positive $x \in U$ *satisfies* $\lim_{t\to\infty} x(t) = 0$.

5. POSSIBLE EXTENSIONS. EXAMPLES

The fact that n is even did not play an important role in the proofs of the theorems, so analogous results are expected to hold for n odd. The results of this paper can be naturally extended to equations of the from

$$x^{(n)} + P(t, x) = Q(t, x),$$

where $P_1(t)\,G_1(x) \leqslant P(t, x) \leqslant P_2(t)\,G_2(x)$ under assumptions on P_i and G_i similar to those on P and G. There is no essential difficulty in transferring these results to equations with deviating arguments of the form

$$x^{(n)} + P(t, x(t), x(g(t))) = Q(t, x(t), x(g(t))),$$

where the function g is of the type considered (for example) by Ladas in [6].

As Hammett notes in [8], it would be interesting to know whether the condition $P(t) \geqslant M/t^k$ in Theorem 5 can be replaced by

$$\int_0^\infty t^k P(t)\,dt = +\infty.$$

It is easy to find examples of equations satisfying the hypotheses of Theorems 1, 3, or 4. In Theorem 2 no solution belongs to class II if $P = P_1 + P_2$, $P_1(t) > 0$,

$$\int_0^\infty s^k P_1(s)\,ds = +\infty, \qquad \int_0^\infty s^k P_2(s)\,ds < +\infty,$$

$L_1 \leqslant G(x) \leqslant L_2$ for all $x > 0$ (L_1 and L_2 are positive constants), and $|\,Q(t, x)| \leqslant Q_0(t)$ for all (t, x) with $x > 0$ and

$$\int_0^\infty s^k Q(s)\,ds < +\infty.$$

Now we present some examples which illustrate the variety of situations that can arise under the assumptions of Theorem 5.

Example 1. The following equation satisfies all the assumptions of Corollary 1 and possesses unbounded nonoscillatory solutions:

$$x^{(n)} + t^{-n}(t^{-a-1} - a(a-1)(a-2)\cdots(a-(n-1))\,x = t^{-(n+1)},$$

where $n - 2 < a < n - 1$. Solution: $x(t) = t^a$.

Example 2. An equation satisfying the hypotheses of Corollary 1 and having bounded eventually positive solutions is

$$x^{(n)} + (\sin t + t^{-\epsilon})\,t^{-n+1}x^m = (t^{-1})^{(n)} + (t^{-1})^{m+n+1}(\sin t + t^{-\epsilon}),$$

where $t \geqslant 1$, $m =$ an odd integer $\geqslant 3$, $0 < \epsilon < 1$. Here we have the solution $x(t) = t^{-1}$, $t \geqslant 1$.

EXAMPLE 3. Equations satisfying the assumptions of Corollary 1 can have bounded oscillatory solutions; for example,

$$x^{(n)} + (\sin t + \tfrac{1}{2})\, t^{-n} x^3 = [(\sin t + \tfrac{1}{2})\, t^{-r}]^{(n)} + (\sin t + \tfrac{1}{2})^4\, t^{-3r-n},$$

where r is a positive constant $\geqslant n+1$. Here we have the solution $x(t) = (\sin t + \frac{1}{2})\, t^{-r}$.

EXAMPLE 4. There are equations which satisfy the assumptions of Theorem 5 and have eventually positive solutions (which have to satisfy $\lim_{t\to\infty} x(t) = 0$):

$$x^{(n)} + (r/t^k)\, x^5 = (t^{-1})^{(n)} + (r/t^k)\, t^{-5},$$

where r is a positive constant and $0 \leqslant k \leqslant n-2$. Solution: $x(t) = t^{-1}$.

It is obvious that the study of positive solutions of $(*)$ is intimately related to the study of the existence of oscillatory solutions. It is hoped that the results of this paper can be used in this direction. For some recent oscillation results concerning the case of a noneventually positive P, and $Q \equiv 0$, the reader is referred to [1–3, 6, 7].

REFERENCES

1. N. P. BHATIA, Some oscillation theorems for second order differential equations, *J. Math. Anal. Appl.* **15** (1966), 442–446.
2. W. J. COLES, Oscillation criteria for nonlinear second order equations, *Ann. Mat. Pura Appl.* **82** (1969), 123–134.
3. A. G. KARTSATOS, Contributions to the research of the oscillation and the asymptotic behaviour of solutions of ordinary differential equations, *Bull. Soc. Math. Grèce* **10** (1969), 1–48.
4. A. G. KARTSATOS, On oscillation of solutions of even order nonlinear differential equations, *J. Differential Equations* **6** (1969), 232–237.
5. A. G. KARTSATOS, On the maintenance of oscillations of nth order equations under the effect of a small forcing term, *J. Differential Equations* **10** (1971), 355–363.
6. G. LADAS, Oscillation and asymptotic behaviour of solutions of differential equations with retarded arguments, *J. Differential Equations* **10** (1971), 281–290.
7. C. TRAVIS, A note on second order nonlinear oscillations, *Math. Japon.*, in press.
8. M. HAMMETT, Nonoscillation properties of a nonlinear differential equation, *Proc. Amer. Math. Soc.* **30** (1971), 92–96.